Goldblatt - Thomason Theorem for Łukasiewicz Finitely-Valued Modal Language

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1 Introduction and overview

Propositional modal logic is often advertised as being a way to talk about relational structures and conversely. One can indeed consider two types of problems depending on what one wants to focus is attention on.

On the one hand, if one is interested in the deductive aspects of the modal language, then one can study relational semantics in order to build completeness results. On the other hand, if one is interested in the descriptive power of the modal language, then one can try to characterize classes of relational structures that are modally definable.

We are interested in languages $\mathcal{L}$ that are modal extensions of the language of Łukasiewicz logic (it means that connectives $\neg$ and $\rightarrow$ are intended to be interpreted in a Łukasiewicz way). Several authors have considered the deductive aspects of such languages ([1,2,3,4]). Among the crisp structures, it turned out that there are two classes of relational structures that are relevant to interpret these languages. The first one is the class of $\mathcal{L}$-frames and the second one is the class of L_n-valued $\mathcal{L}$-frames. The latter are Kripke frames in which the set of allowed truth values is specified in each world of the frame. These two classes of structures give rise to two different notions of Kripke completeness ([4]).

In this talk, we study the descriptive power of such languages $\mathcal{L}$ with regards to these two types of relational structures. More precisely, we give many-valued generalizations of the celebrated Goldblatt – Thomason characterization of modally definable classes of Kripke frames that are closed under ultrapowers ([5]).

Hence, our two main results are the following. They involve new notions that are introduced in the remainder of the paper.

Theorem 1. *Assume that $\mathcal{C}$ is a class of Ł$_n$-valued $\mathcal{L}$-frames that contains ultrapowers of its elements. Then $\mathcal{C}$ is definable if and only if the following two conditions are satisfied.*

1. *The class $\mathcal{C}$ contains Ł$_n$-valued generated subframes, disjoint unions and Ł$_n$-valued bounded morphic images of its members.*
2. *For any Ł$_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, if $\mathfrak{Ce}(\mathfrak{F}) \in \mathcal{C}$ then $\mathfrak{F} \in \mathcal{C}$.*

Theorem 2. *Assume that $\mathcal{C}$ is a class of $\mathcal{L}$-frames that contains ultrapowers of its elements. Then $\mathcal{C}$ is $Ł_n$-definable if and only if the following two conditions are satisfied.*

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2 Language and notations

Let $\mathcal{L} = \{\neg, \rightarrow, 0\} \cup \{\nabla_i \mid i \in I\}$ be a language, where $\neg$ is unary, $\rightarrow$ is binary, 0 is constant and ∇_i is n_i-ary for any $i \in I$. The set $\mathsf{Form}_\mathcal{L}$ of formulas is defined by induction from a countably infinite set of propositional variables Prop using the grammar

$$\phi ::= p \in \mathsf{Prop} \mid 0 \mid \neg\phi \mid \phi \rightarrow \phi \mid \nabla_i(\phi, \dots, \phi).$$

Elements of $\{\nabla_i \mid i \in I\}$ are called a *modalities* (our modalities are universal ones). We sometimes write $\phi(p_1, \dots, p_k)$ to stress that ϕ is a formula whose propositional variables are among $p_1, \dots, p_k$.

We use bold letters to denote tuples (arity is given by the context). Hence, we denote by $\boldsymbol{\phi}, \boldsymbol{\psi}, \dots$ tuples of formulas and by ϕ_i the ith component of $\boldsymbol{\phi}$. If $R \subseteq W^n$, we write $\mathbf{u} \in R$ for $(u_1, \dots, u_n) \in R$ and $\mathbf{w} \in Ru$ for $(u, w_1, \dots, w_{n-1}) \in R$.

We use standard abbreviations: $\phi \oplus \psi$ stands for $\neg\phi \rightarrow \psi$, $\phi \odot \psi$ stands for $\neg(\neg\phi \oplus \neg\psi)$, $\phi \vee \psi$ stands for $(\psi \odot \neg\phi) \oplus \phi$, $\phi \wedge \psi$ stands for $(\psi \oplus \neg\phi) \odot \phi$, if k is an nonnegative integer then ϕ^k stands for $\phi \odot \cdots \odot \phi$ (with k factors ϕ).

We use n to denote a positive integer and $Ł_n$ to denote the sub-MV-algebra $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ of the standard MV-algebra $[0, 1]$.

3 $\mathcal{L}$-frames and $Ł_n$-valued $\mathcal{L}$-frames

Definition 1 ($\mathcal{L}$-frame, $Ł_n$-model). *An $\mathcal{L}$-frame, is a tuple $(W, (R_i)_{i \in I})$ where W is a nonempty set and R_i is an $n_i + 1$-ary relation for any $i \in I$. Elements of the set W are called worlds. We denote by $\mathcal{FR}_\mathcal{L}$ the class of $\mathcal{L}$-frames.*

An $Ł_n$-model is a couple $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ where $\mathfrak{F} = (W, (R_i)_{i \in I})$ is an $\mathcal{L}$-frame and $\mathrm{Val} : W \times \mathsf{Prop} \rightarrow Ł_n$. We say that $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ is based on $\mathfrak{F}$.

In an $Ł_n$-model $\mathcal{M}$, the valuation map Val is extended inductively to $\mathsf{Form}_\mathcal{L}$ using Łukasiewicz interpretation of the connectors $\neg$ and $\rightarrow$ in $[0, 1]$ and the rules $\mathrm{Val}(u, \nabla_i(\boldsymbol{\phi})) = \bigwedge\{\bigvee_{1 \leq k \leq n_i} \mathrm{Val}(w_k, \phi_k) \mid \mathbf{w} \in Ru\}$ for any $i \in I$.

Definition 2 (True, $Ł_n$-valid). *If $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ is an L_n-model and if $\phi \in \mathrm{Form}_\mathcal{L}$, we note $\mathcal{M} \models_n \phi$ if $\mathrm{Val}(u, \phi) = 1$ for any world u of $\mathfrak{F}$. We say that ϕ is true in $\mathcal{M}$.*

If Φ is a set of $\mathcal{L}$-formulas that are true in any $Ł_n$-model based on a frame $\mathfrak{F}$, we write $\mathfrak{F} \models_n \Phi$ and say that Φ is L_n-valid in $\mathfrak{F}$. We write $\mathfrak{F} \models_n \phi$ instead of $\mathfrak{F} \models_n \{\phi\}$.

Definition 3 ($Ł_n$-definable). *A class $\mathcal{C}$ of $\mathcal{L}$-frames is $Ł_n$-definable if there is a $\Phi \subseteq \mathsf{Form}_{\mathcal{L}}$ such that $\mathcal{C} = \{\mathcal{F} \in \mathcal{FR}_{\mathcal{L}} \mid \mathfrak{F} \models_n \Phi\}$. In that case, we write $\mathcal{C} = \mathrm{Mod}_n(\Phi)$.*

We denote by $\mathsf{PForm}^n_{\mathcal{L}}$ the fragment of $\mathsf{Form}_{\mathcal{L}}$ defined by the grammar $\phi ::= p^n \mid 0 \mid \neg\phi \mid \phi \to \phi \mid \nabla_i(\phi, \dots, \phi)$ where $p \in \mathrm{Prop}$ and $i \in I$.

Let tr_n be the map $\mathrm{tr}_n : \mathsf{Form}_{\mathcal{L}} \to \mathsf{PForm}^n_{\mathcal{L}} : \phi(p_1, \dots, p_k) \mapsto \phi(p_1^n, \dots, p_k^n)$.

Lemma 1. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-frames and $\Phi \subseteq \mathrm{Form}_{\mathcal{L}}$. The following conditions are equivalent.*

1. *$\mathcal{C} = \mathrm{Mod}_1(\Phi)$.*
2. *There is an $n > 0$ such that $\mathcal{C} = \mathrm{Mod}_n(\mathrm{tr}_n(\Phi))$.*
3. *For any $n > 0$, $\mathcal{C} = \mathrm{Mod}_n(\mathrm{tr}_n(\Phi))$.*

Moreover $\mathrm{Mod}_n(\Phi) \subseteq \mathrm{Mod}_1(\Phi)$ for any $n > 0$.

Next example illustrates that $\mathrm{Mod}_n(\Phi)$ may differ from $\mathrm{Mod}_1(\Phi)$.

Example 1. Let $\mathcal{L}_\Box$ be the modal language with a single unary modality $\Box$ and $n > 1$. Then $\mathrm{Mod}_1(\Box(p \vee \neg p)) = \mathcal{FR}_{\mathcal{L}_\Box}$ while $\mathrm{Mod}_n(\Box(p \vee \neg p)) = \{(W, R) \in \mathcal{FR}_{\mathcal{L}_\Box} \mid R = \varnothing\}$.

For any positive integer n, we denote by $\mathrm{div}(n)$ the set of its positive divisors.

Definition 4 ($Ł_n$-valued $\mathcal{L}$-frame). *An $Ł_n$-valued $\mathcal{L}$-frame is a tuple $(W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i \in I})$ where*

1. *$(X, (R_i)_{i \in I})$ is an $\mathcal{L}$-frame,*
2. *$r_m \subseteq W$ for any $m \in \mathrm{div}(n)$,*
3. *$r_n = W$ and $r_m \cap r_q = r_{\gcd(m,q)}$ for any $m, q \in \mathrm{div}(n)$,*
4. *$R_i u \subseteq r_m^{n_i}$ for any $i \in I$, any $m \in \mathrm{div}(n)$ and any $u \in r_m$.*

We denote by $\mathcal{F}_\sharp$ the underlying $\mathcal{L}$-frame of the $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ and by $\mathcal{FR}^n_{\mathcal{L}}$ the class of the $Ł_n$-valued $\mathcal{L}$-frames.

The trivial $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}^n_b$ associated to an $\mathcal{L}$-frame $\mathfrak{F}$ is obtained by enriching $\mathfrak{F}$ with $\{r_m \mid m \in \mathrm{div}(n)\}$ where $r_m = \varnothing$ if $m \neq n$ and $r_n = W$.

As explained in the next definition, the structure given by the sets r_m (where $m \in \mathrm{div}(n)$) is used to weaken the validity relation.

Definition 5 (Validity in $Ł_n$-valued $\mathcal{L}$-frames). *An $Ł_n$-model $\mathcal{M} = (\mathfrak{F}', \mathrm{Val})$ is based on the $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F} = (W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i \in I})$ if $\mathfrak{F}' = \mathfrak{F}_\sharp$ and $\mathrm{Val}(u, \mathsf{Prop}) \subseteq Ł_m$ for any $m \in \mathrm{div}(n)$ and any $u \in r_m$.*

If Φ is a set of $\mathcal{L}$-formulas that are true in any $Ł_n$-model based on a $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, we write $\mathfrak{F} \models \Phi$ and say that Φ is valid in $\mathfrak{F}$. We write $\mathfrak{F} \models \phi$ instead of $\mathfrak{F} \models \{\phi\}$.

Definition 6 (Definability). *A class $\mathcal{C}$ of $Ł_n$-valued $\mathcal{L}$-frames is definable if there is a $\Phi \subseteq \mathrm{Form}_{\mathcal{L}}$ such that $\mathcal{C} = \{\mathcal{F} \in \mathcal{FR}^n_{\mathcal{L}} \mid \mathfrak{F} \models \Phi\}$. In that case, we write $\mathcal{C} = \mathrm{Mod}(\Phi)$.*

Example 2. One can check that $\mathrm{Mod}(\Box(p \vee \neg p)) = \{\mathfrak{F} \in \mathcal{FR}^n_{\mathcal{L}_\Box} \mid \forall u\, Ru \subseteq r_1\}$. Moreover $\{\mathfrak{F} \in \mathcal{FR}^n_{\mathcal{L}_\Box} \mid \forall u\, u \notin r_m\}$ is not definable if m is a strict divisor of n.

4 $Ł_n$-valued $\mathcal{L}$-frame constructions

$\mathcal{L}$-frame constructions used in the statement of Theorem 2 are classical in modal logic (see [6] for example). We define the $Ł_n$-valued $\mathcal{L}$-frame constructions needed to understand statement of Theorem 1.

Definition 7 ($Ł_n$-valued bounded morphism). *A map $f : \mathfrak{F} \to \mathfrak{F}'$ between two $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F} = (W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i\in I})$ and $\mathfrak{F}' = (W', \{r'_m \mid m \in \mathrm{div}(n)\}, (R'_i)_{i\in I})$ is an $Ł_n$-valued bounded morphism if f is a bounded morphism between $\mathfrak{F}_\sharp$ and $\mathfrak{F}'_\sharp$ and if $f(r_m) \subseteq r'_m$ for any $m \in \mathrm{div}(n)$.*

Definition 8 ($Ł_n$-valued generated subframe). *An $\mathcal{L}$-substructure $\mathfrak{F}'$ of an $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ is called an $Ł_n$-valued generated subframe of $\mathfrak{F}$ if the inclusion map $\iota : \mathfrak{F}' \to \mathfrak{F}$ is an $Ł_n$-valued bounded morphism.*

If u is a world of an $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, we denote by s_u the integer $\gcd\{m \in \mathrm{div}(n) \mid u \in r_m\}$.

Definition 9 (Canonical extension). *Let $\mathfrak{F} = (W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i\in I})$ be an $Ł_n$-valued $\mathcal{L}$-frame. We denote by $\mathfrak{F}_\times$ the $\mathcal{L}$-algebra whose universe is $\prod_{u\in W} Ł_{s_u}$ with operations 0, $\neg$ and $\to$ defined pointwise and ∇_i defined by*

$$\nabla_i(\boldsymbol{\alpha})(u) = \bigwedge_{\mathbf{w}\in R_i u} \bigvee_{1\le k\le n_i} \alpha_k(w_k),$$

for any $i \in I$.

The canonical extension of $\mathfrak{F}$, in notation $\mathfrak{Ce}(\mathfrak{F})$ is the structure $(W^e, \{r^e_m \mid m \in \mathrm{div}(n)\}, (R^e_i)_{i\in I})$ defined by:

- *$W^e = \mathcal{MV}(\mathfrak{F}_\times, Ł_n)$ is the set of MV-homomorphisms from $\mathfrak{F}_\times$ to $Ł_n$,*
- *$u \in r^e_m$ if $u(\mathfrak{F}_\times) \subseteq Ł_m$,*
- *$(u, \mathbf{w}) \in R^e_i$ if $\bigvee_{1\le k\le n_i} w_k(\alpha_k) = 1$ for any $\boldsymbol{\alpha} \in \mathfrak{F}_\times^{n_i}$ such that $u(\nabla_i\boldsymbol{\alpha}) = 1$.*

If $\mathfrak{F}$ is an $\mathcal{L}$-frame, the canonical extension of $\mathfrak{F}$, in notation $\mathfrak{Ce}(\mathfrak{F})$, is the $\mathcal{L}$-frame $(\mathfrak{Ce}(\mathfrak{F}^1_b))_\sharp$.

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